Como enxergar "a coleção de setas entre X & Y em $\mathscr{C}$" como funtor?

<u>Notação</u>: Dados categoria $\mathscr{C}$, objetos X, Y de $\mathscr{C}$, denotamos por

$$\mathscr{C}(X, Y)$$

ou $\mathrm{Hom}_{\mathscr{C}}(X, Y)$

a coleção de setas de $\mathscr{C}$ com dom X e cod Y.

Varrido para baixo do tapete: qual a "natureza" de $\mathcal{C}(X,Y)$? Em geral não é conjunto, mas vamos fingir que sim :)

— x —

- $\mathcal{C}(X,-) : \mathcal{C} \to \mathrm{Set}$

$$Y \in \mathcal{O}(\mathcal{C}) \longmapsto \mathcal{C}(X,Y)$$

$$f : Y \xrightarrow[\mathcal{C}]{} Z \in A(\mathcal{C}) \longmapsto \mathcal{C}(X, f) : \mathcal{C}(X,Y) \xrightarrow{\mathrm{Set}} \mathcal{C}(X,Z)$$

$$\mathcal{C}(X,f)\left(g : X \xrightarrow[\mathcal{C}]{} Y\right) = g;f$$

$\mathcal{C}$

$$\begin{tikzcd} X \arrow[dr, bend left, "g;f"] \arrow[d, bend right, "g"'] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

$$\mathcal{C}(X,f) = -;f$$

Preservação de compostas:

A provar: $\mathscr{C}(X, f;_{\mathscr{C}} g) = \underbrace{\mathscr{C}(X, f)}_{(-;_{\mathscr{C}} f)\ =\ \lambda h.\, h;_{\mathscr{C}} f} ;_{Set} \underbrace{\mathscr{C}(X, g)}_{(-;_{\mathscr{C}} g)\ =\ \lambda k.\, k;_{\mathscr{C}} g}$

$$\mathscr{C}(X, f;_{\mathscr{C}} g) = -;_{\mathscr{C}}(f;_{\mathscr{C}} g)$$

$$(-;_{\mathscr{C}} f) ;_{Set} (-;_{\mathscr{C}} g)$$

$$(- \,;_{\mathcal{C}}\, f) \;;_{Set}\; (- \,;_{\mathcal{C}}\, g) = \lambda l.\ (l \,;_{\mathcal{C}}\, f) \,;_{\mathcal{C}}\, g$$

$$= (- \,;_{\mathcal{C}}\, f) \,;_{\mathcal{C}}\, g$$

$=$ porque

$;_{\mathcal{C}}$ é associativa!!!

- $\mathcal{C}(-,Y) : \mathcal{C} \to Set$

$$X \in O(\mathcal{C}) \longmapsto \mathcal{C}(X,Y)$$

$$f : X \xrightarrow[\mathcal{C}]{} Z \in A(\mathcal{C}) \longmapsto \mathcal{C}(f,Y) : \mathcal{C}(X,Y) \xrightarrow{set} \mathcal{C}(Z,Y)$$

Em $\mathcal{C}$

$$\begin{tikzcd} & Y \\ X \arrow[ur, "g"] \arrow[r, "f"'] & Z \end{tikzcd}$$

DEU RUIM!

- $\mathcal{C}(-,Y) : \mathcal{C}^{op} \to Set$

  $X \in U(\mathcal{C}) \longmapsto \mathcal{C}(X,Y)$

  $f : X \xrightarrow[\mathcal{C}^{op}]{} Z \in A(\mathcal{C}^{op}) \longmapsto \mathcal{C}(f,Y) : \mathcal{C}(X,Y) \xrightarrow{Set} \mathcal{C}(Z,Y)$

Em $\mathcal{C}$

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "g"] & & Z \arrow[ll, "f"] \arrow[ul, dashed, "f ;_{\mathcal{C}} g"'] \end{tikzcd}$$

$\mathcal{C}(f,Y) = f ;_{\mathcal{C}} -$

DEU BOM!.

:)

$$\mathcal{C}(X, -) : \mathcal{C} \to \mathrm{Set}$$, "pós- compor"

$$\mathcal{C}(-, Y) : \mathcal{C}^{op} \to \mathrm{Set}$$, "pré-compor"

$$\mathcal{C}(-_0, -_1) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$$

def v3: Sejam $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$,

Então $F \dashv G$ se

$$\mathcal{C}(-_0, G(-_1)) \cong \mathcal{D}(F(-_0), -_1)$$

Note que ambos os funtores são $\mathcal{C}^{op} \times \mathcal{D} \longrightarrow Set$

Parece que $\mathcal{C}(-_0, G(-_1))$

$$= (id(\mathcal{C}^{op}) \times G) \,;\, \mathcal{C}(-_0, -_1)$$

em qual categoria?

Cat!

e que

$\mathcal{D}(F(-_0), -_1)$

$$= ((op \,;\, F) \times id) \,;\, \mathcal{D}(-_0, -_1)$$

Na última aula, buscamos $F \dashv X$

No novo formato, queremos $F : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ tal que

$$\mathscr{C}(-_0, (-_1 \times -_2)) \simeq \mathscr{C} \times \mathscr{C}(F(-_0), (-_1, -_2))$$

Plugando objetos $A, B, C \in \mathcal{O}(\mathscr{C}) = \mathcal{O}(\mathscr{C}^{op})$

$$\mathcal{C}(A, B\times C) \simeq \mathcal{C}\times\mathcal{C}\big(F(A), (B,C)\big)$$

$$\simeq \left(\begin{array}{c} \mathcal{C}(F_0(A), B) \\ \times \\ \mathcal{C}(F_1(A), C) \end{array} \right)$$

$$\begin{tikzcd}
 & A \arrow[d, "f"] \arrow[dl, "f;\pi_0"'] \\
B & B\times C \arrow[l, "\pi_0"]
\end{tikzcd}$$

Chute informado: $F(A) = (A, A)$

E nas setas?

$$f : A \xrightarrow[\mathcal{C}^{op}]{} B \,, \qquad g : C \xrightarrow[\mathcal{D}]{} D \qquad\qquad g \times h : C \times C' \xrightarrow[\mathcal{C}]{} D \times D'$$

$$h : C' \xrightarrow[\mathcal{D}]{} D'$$

$$\text{Então } F(f) : F(B) \to F(A)$$

$$\text{Quero: } \quad \begin{array}{c} F(B) \\ \| \\ (B, B) \end{array} \to (A, A)$$

$$\mathcal{C}(f, g \times h) \simeq \mathcal{C} \times \mathcal{C}\,(F(f), (g, h))$$

$$\|$$

$$(F_0 f \,;\, -_0 \,;\, g,\; F_1(f) \,;\, -_1 \,;\, h) = F(f) \;;_{\mathcal{C} \times \mathcal{C}}\; (-_0, -_1) \;;_{\mathcal{C} \times \mathcal{C}}\; (g, h)$$

$$f ; -_3 ; (g \times h)$$

$$=$$

$$\left\langle f ; -_3 ; \pi_0 ; g ,\ f ; -_3 ; \pi_1 ; h \right\rangle$$

$$\begin{tikzcd}
C \arrow[d, "g"'] & C \times C' \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, dashed, "{\langle \pi_0 ; g ,\ \pi_1 ; h \rangle}"] & C' \arrow[d, "h"] \\
D & D \times D' \arrow[l] \arrow[r] & D'
\end{tikzcd}$$

Queria : $F(f) = (f, f)$

F é o funtor cópia!

Denotado Δ "diagonal"

Quando $\mathcal{C}$ tem produtos, temos

$$\Delta \dashv \times$$

Quando tem coprodutos, $+ \dashv \Delta$

Muitas vezes $\times$ tem adj à direita: exponencial!

$$+ \dashv \Delta \dashv \times \dashv \exp$$

Para refletir: repetir com id no lugar de Δ. Quem seriam $+$ & $\times$?