$$\begin{tikzcd} \mathcal{E} \arrow[r, "H"] & \mathcal{C} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, "K"] & \mathcal{J} \end{tikzcd}$$

Quero uma transf. nat.

$$\varepsilon : F ; K \to G ; K$$

Dados: $\varepsilon : \mathcal{U}(\mathcal{C}) \to \Delta(\mathcal{J})$

Notação

$$\eta \triangleright K$$

$K\eta$ (livros)

$$X \in \mathcal{U}(\mathcal{C}) \longmapsto K(\eta(X))$$

a) Assinatura: $\forall X\,(\mathcal{U}(\mathcal{C}))$ tenho que ter

$$\varepsilon(X) = K(\eta(X))$$

$$\varepsilon(X) : (F;K)(X) \to (G;K)(X)$$

Sei sobre $\eta : F \xrightarrow[\text{Fun}]{} G$:

$$\eta(X) : F(X) \xrightarrow[\mathcal{D}]{} G(X)$$

Logo $K(\eta(X)) : K(F(X)) \xrightarrow[\mathcal{J}]{} K(G(X))$

ie, $\varepsilon(X) : (F;K)(X) \to (G;K)(X)$

b) Naturalidade: "o quadrado comuta"

Provar: $\forall f : X \to Y \in \mathcal{A}(\mathcal{C})$

$$\begin{tikzcd}
(F;K)(X) \arrow[r, "(F;K)(f)"] \arrow[d, "\varepsilon(X)"'] & (F;K)(Y) \arrow[d, "\varepsilon(Y)"] \\
(G;K)(X) \arrow[r, "(G;K)(f)"'] & (G;K)(Y)
\end{tikzcd}$$

comuta

$$\begin{tikzcd}
K(F(X)) \arrow[r, "K(F(f))"] \arrow[d, "K(\eta(X))"'] & K(F(Y)) \arrow[d, "K(\eta(Y))"] \\
K(G(X)) \arrow[r, "K(G(f))"'] & K(G(Y))
\end{tikzcd}$$

em $\mathcal{J}$

mas :

$$
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta(X)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\eta(Y)"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}
$$

Comuta, pois η é natural! e K preserva isso por ser funtor

$$
\begin{aligned}
& K(F(f)) \,;_{g}\, K(\eta(Y)) \\
\overset{\text{fun}}{=}\; & K(F(f) \,;_{\mathcal{D}}\, \eta(Y)) \\
\overset{\text{nat}}{=}\; & K(\eta(X) \,;_{\mathcal{D}}\, G(f)) \\
\overset{\text{fun}}{=}\; & K(\eta(X)) \,;_{g}\, K(G(f))
\end{aligned}
$$

A transf. natural da última aula pode ser derivada

$$H \triangleright \eta : H;F \to H;G$$

$$\eta_H \quad (\text{livros})$$

— x —

Rascunho

$$
\begin{tikzcd}
\mathcal{C} \arrow[loop left, "id(\mathcal{C})"] \arrow[rr, bend left=50, "F"] & & \mathcal{D} \arrow[ll, bend left=50, "G"] \arrow[loop right, "id(\mathcal{D})"]
\end{tikzcd}
$$

$$\eta : F;G \xrightarrow[\text{Fun}]{} id(\mathcal{C}) \qquad\qquad \varepsilon : id(\mathcal{D}) \xrightarrow[\text{Fun}]{} G;F$$

$\eta \rhd F : F;G;F \longrightarrow F$ | $G \rhd \eta : G;F;G \to G$

$\varepsilon \rhd G : G \longrightarrow G;F;G$ | $F \rhd \varepsilon : F \to F;G;F$

Errado!
ao contrário

def: Dados $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$

$F \dashv G$ sse $\exists \eta : F;G \to id(\mathcal{C})$

$\exists \varepsilon : id(\mathcal{D}) \to G;F$

"F é adjunto à esquerda de G"

satisfazendo $\begin{cases} (F \triangleright \varepsilon) ; (\eta \triangleright F) = id(F) \\ (\varepsilon \triangleright G) ; (G \triangleright \eta) = id(G) \end{cases}$

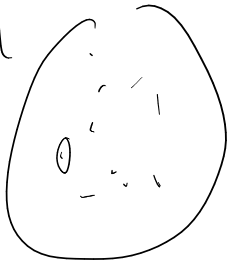
η é chamado "unidade" da adjunção

ε é chamado "counidade" da adjunção

— x —

"Livre"

M



$e \qquad x$

$\parallel$

x^2

e

$e \quad x \quad x^2 \quad x^3 \quad x^4 \quad \ldots$

$e \quad \cdot x \quad \cdot x^2 \quad \ldots$

$\quad \cdot y \quad \cdot y^2 \quad \ldots$

$\quad \cdot xy$

} palavras usando apenas x, y como letras

No geral, como vimos, dado um conjunto X, o monoide livremente gerado por X é dado por

· Universo: palavras finitas cujas letras são elem. de X

· operação: Concatenação

· neutro: palavra vazia.

Homomorfismos

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \end{tikzcd}$$

livre

$$\begin{tikzcd}[column sep=huge]
\text{Set} \arrow[r, bend left, "\text{Livre}"] & \text{Monoides} \arrow[l, bend left, "\text{Esquecedor}"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
X \arrow[d, "f"] & \text{Livre}(X) \arrow[d, "\text{Livre}(f)"] \\
Y & \text{Livre}(Y)
\end{tikzcd}$$

$$x_0 \, x_1 \, x_2 \rightsquigarrow f(x_0) \, f(x_1) \, f(x_2)$$

"O monoide mais livre possível que contém X"

Para qualquer monoide M que "contenha" X, existe único homomorfismo

$\varphi : \text{Livre}(X) \to M$ "preservando" X, i.e., tal que $\varphi(x) = x \quad \forall x \in X$

<u>def</u> (Monoide livremente gerado por X)

Livre(X) é especificado por:

$\exists\, \hat{\imath} : id(Set) \to \text{Livre} ; \text{Esq}$

$\forall$ monoide M

$\forall f : X \to_{Set} \text{Esq}(M)$

$\exists!\, \varphi : \text{Livre}(X) \to_{Mon} M$

tal que

$$f = \hat{\imath}(X) ; \text{Esq}(\varphi)$$

$$\begin{tikzcd} Set \arrow[r, shift left, "Livre"] & Monoides \arrow[l, shift left, "Esq"] \end{tikzcd}$$

$\text{Esq}(\varphi) : \text{Esq}(\text{Livre}(X)) \to \text{Esq}(M)$

$\text{Livre} \dashv \text{Esq}$

Set

$$\begin{tikzcd}
\mathrm{Esq}(\mathrm{Livre}(X)) \arrow[dr, "\mathrm{Esq}(\varphi)"] \arrow[d, phantom, shift left=6ex, "\circlearrowright"] & \\
X \arrow[u, "\exists_0 i(X)"] \arrow[r, "\forall_2 f"'] & \mathrm{Esq}(\mathcal{M})
\end{tikzcd}$$

Monoides

$$\mathrm{Livre}(X) \xrightarrow{\exists !_3 \varphi} \forall_1 \mathcal{M}$$